Planarity via Spanning Tree Number
A Linear-Algebraic Criterion

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Highlights

- We define a quantity "**excess**" $\varepsilon(G)$ for any undirected (multi)graph G in terms of the number of spanning trees and a linear-algebraic optimization problem, which satisfies
$$\begin{aligned} &\varepsilon(G) = 0, && \text{if } G \text{ is planar,} \\ &\varepsilon(G) \geq 18, && \text{if } G \text{ is nonplanar.} \end{aligned}$$
- This characterization gives a **certificate of planarity** that can be easily verified by computing the determinant of a sparse matrix and counting spanning trees.
- We show that any **subdivision of** $K_{3,3}$ **or** K_5, two important nonplanar graphs, **underperforms** the best planar graph with the same number of edges in some linear-algebraic sense.
- Our linear-algebraic connection gives an upper bound on **the maximum number of spanning trees in a planar (multi)graph with a fixed number of edges**, which matches the current best upper bound.

Definitions

Given a matrix M,

- we say that M is an **incidence submatrix** if each row of M has at most one 1, at most one -1, and all other entries 0.

Given a connected graph G with an orientation D,

- we use $\tau(G)$ to denote the number of spanning trees in G;
- a **truncated incidence matrix** trun(D) of G is the incidence matrix of D with an arbitrary column removed;
- let maxdet(G) be the maximum determinant of a square concatenation $[M|N]$ such that M is a truncated incidence matrix of G and N is an incidence submatrix;
- define the **excess** of G to be $\varepsilon(G) := \tau(G) - \text{maxdet}(G)$.

Proposition 1: Excess is nonnegative

For any connected graph G, we have $\varepsilon(G) \geq 0$.

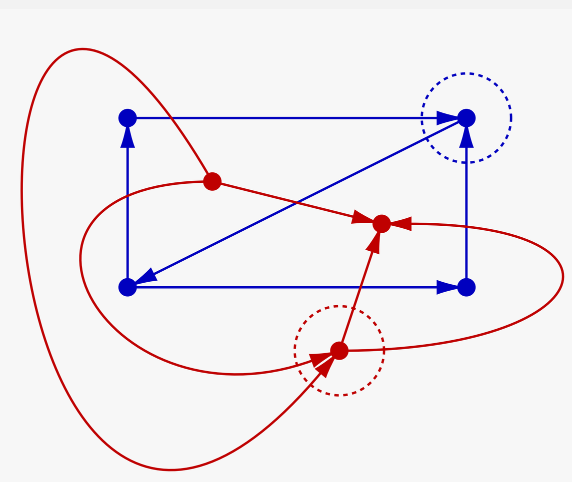
Theorem 2: Planarity criterion via excess

For any connected graph G,
$$\begin{aligned} &\varepsilon(G) = 0, && \text{if } G \text{ is planar,} \\ &\varepsilon(G) \geq 18, && \text{if } G \text{ is nonplanar.} \end{aligned}$$

Lemma 3: Excess is zero for planar graphs

Let G be a connected planar graph. Let D be an orientation of G with directed planar dual D^*. Then
$$\left|\det\left[\text{trun}(D) \middle| \text{trun}(D^*)\right]\right| = \tau(G),$$
where the i^{th} rows of trun(D) and trun(D^*) correspond to the same arc.

A planar example



(a) A connected planar digraph and its directed planar dual, each with 5 edges, where the two circled vertices are the ones truncated in their truncated incidence matrices, respectively.

$$\left[\begin{array}{ccc|cc} -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & -1 \end{array}\right]$$

(b) A 5×5 matrix whose determinant has absolute value equal to the number of spanning trees in the underlying undirected graph.

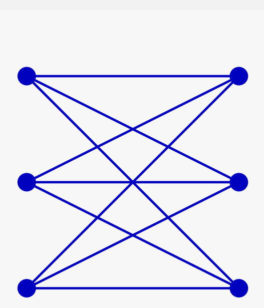
Lemma 4: Merge-cut lemma

For any connected graph $G = (V, E)$ and non-bridge $e \in E$, $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e)$.

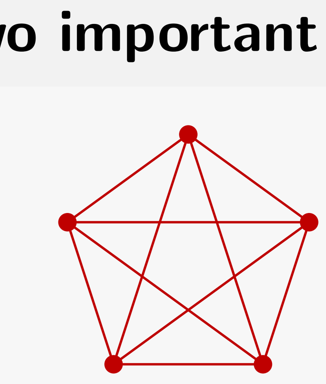
Lemma 5: Excess is at least 18 for nonplanar graphs

For any connected nonplanar graph G, we have $\varepsilon(G) \geq 18$.

Two important nonplanar graphs



(a) $\varepsilon(K_{3,3}) = 18$.



(b) $\varepsilon(K_5) = 25$.

Theorem 6: Wagner

A graph is nonplanar if and only if it can produce either $K_{3,3}$ or K_5 by a sequence of edge contractions and deletions.

Definitions

Given a matrix P,

- we say that P is a **bi-incidence matrix** if P is the concatenation $[M|N]$ of two incidence submatrices.

Given $m \in \mathbb{N}$,

- let τ_m be the maximum number of spanning trees in a **planar** graph with m edges;
- let Δ_m be the maximum determinant of an $m \times m$ bi-incidence matrix.

Theorem 7: Upper bound

For all $m \in \mathbb{N}$, we have $\tau_m \leq \Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of the equation $x^3 - x^2 - x - 1 = 0$.

Remarks on Theorem 7

- The question of determining the values of τ_m was initially asked by [Kenyon 1996]; a lower bound of 1.7916^m is known, achieved by **square grid graphs**.
- The second inequality $\Delta_m \leq \delta^m$ can be proved by noting that, w.l.o.g., any square bi-incidence matrix has **a column with at most three nonzero entries**, and by **multilinearity of determinants**. The proof is inductive and uses the recurrence relation $\Delta_m \leq \Delta_{m-1} + \Delta_{m-2} + \Delta_{m-3}$.
- This matches the current best upper bound by [Stoimenow 2007], who used a **knot-theoretic** argument.

m	1	2	3	4	5	6	7	8	9	10
τ_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

Table 1. τ_m and Δ_m for $m = 1, \ldots, 10$.

Conjecture 8

For all $m \in \mathbb{N}$, we have $\tau_m = \Delta_m$.

Conjecture 9: Nonplanar graphs underperform planar graphs

For any connected nonplanar graph with m edges, we have maxdet$(G) \leq \tau_m$.

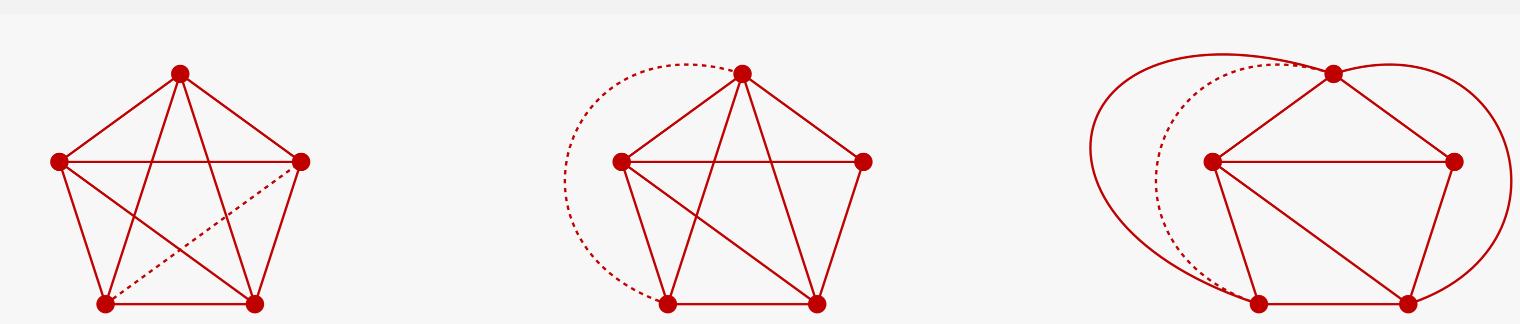
Proposition 10

Conjecture 9 implies Conjecture 8.

Theorem 11: Subdiv. of $K_{3,3}$ and K_5 underperform planar graphs

For any subdivision G of $K_{3,3}$ or K_5 with m edges, we have maxdet$(G) \leq \tau_m$.

Edge relocation method for subdivisions of K_5



Relocating one edge in K_5 to coincide with another edge results in a planar graph.

Future directions

- Does Conjecture 8 hold? What are the exact values of τ_m and Δ_m asymptotically?
- Can the observation that many nonplanar graphs contain several copies of $K_{3,3}$ and K_5 as minors be exploited to strengthen Lemma 5?
- Can the edge relocation method be generalized to a broader class of nonplanar graphs?
- Faster algorithms for counting spanning trees and testing planarity?

Acknowledgments

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References

- Kenyon, R. (1996). "Tiling a rectangle with the fewest squares". In: *Journal of Combinatorial Theory, Series A* 76.2, pp. 272–291.
- Stoimenow, A. (2007). "Maximal determinant knots". In: *Tokyo Journal of Mathematics* 30.1, pp. 73–97.